On the Instability of Einstein's Spherical World.

By A. S. Eddington, F.R.S.

1. Working in conjunction with Mr. G. C. McVittie, I began some months ago to examine whether Einstein's spherical universe is stable. Before our investigation was complete we learnt of a paper by Abbé G. Lemaître * which gives a remarkably complete solution of the various questions connected with the Einstein and de Sitter cosmogonies. Although not expressly stated, it is at once apparent from his formulæ that the Einstein world is unstable—an important fact which, I think, has not hitherto been appreciated in cosmogonical discussions. Astronomers are deeply interested in these recondite problems owing to their connection with the behaviour of spiral nebulæ; and I desire to review the situation from an astronomical standpoint, although my original hope of contributing some definitely new result has been forestalled by Lemaître's brilliant solution.

Finitude of space depends on a " cosmical constant " λ, which occurs in Einstein's gravitational equations $G_{\mu\nu} = \lambda g_{\mu\nu}$ for empty space. On general philosophical grounds † there can be little doubt that this form of the equations is correct rather than his earlier form $G_{\mu\nu} = 0$; but λ is so small as to be negligible in all but very large scale applications. Except in so far as a value may be suggested by astronomical survey of the extragalactic universe, λ is unknown; or it would be better to say that we do not know the lengths of the objects and standards of our ordinary scale of experience in terms of the natural cosmical unit of length $1/\sqrt{\lambda}$. Besides involving λ, the shape and size of space depend on the amount of matter contained in the universe and the way it is distributed. Naturally space will only be of a perfectly spherical form if the matter (which by Einstein's equations controls the geometry) is uniformly distributed. We confine attention to spherical space, so that, strictly speaking, we should suppose the universe to be filled with matter of uniform density; but, practically, we need only insist on *large scale* uniformity, *i.e.* we suppose that the universe is filled with galaxies which are fairly regularly distributed everywhere.

If, further, it is postulated that this spherical universe is permanent and unchanging, there is but one unique solution, viz. Einstein's universe. For equilibrium, space must have a particular radius and contain a particular amount of mass (determinate in terms of the cosmical constant λ).

Technically, de Sitter's solution is also an equilibrium solution; but it is now realised that this is only because, being an entirely empty world, there is nothing in it to show departure from equilibrium.

2. *Expanding Universes.*—An infinite variety of solutions can be found representing spherical worlds which are not in equilibrium. Whilst remaining spherical they expand or contract, the radius (in

* *Annales de la Société Scientifique de Bruxelles*, **47**A, 49 (April 1927).

† *Nature of the Physical World*, chap. vii.

terms of our ordinary standards which are in a constant, though unknown, relation to the cosmical standard $1/\sqrt{\lambda}$) being a function of the time. In an expanding spherical world the galaxies, since they continue to fill space uniformly, must become further apart as time progresses. Expanding solutions are therefore of astronomical interest as a possible explanation of the observed scattering apart of the spiral nebulæ.*

As in Einstein's solution, the interval corresponding to spherical space and undistorted time is

$$ds^2 = -a^2\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\} + dt^2 \qquad . \qquad (1)$$

where a is the radius of space and (χ, θ, ϕ) are angular co-ordinates †; but we now allow a to be a function of t. Lemaître shows that Einstein's gravitational equations give the following values for the density ρ and pressure p of matter in the space (1).

$$8\pi\rho = -\lambda + 3\left\{\frac{1}{a^2}\left(\frac{da}{dt}\right)^2 + \frac{1}{a^2}\right\} \qquad . \quad . \quad . \qquad (2)$$

$$8\pi p = \lambda - \left\{\frac{1}{a^2}\left(\frac{da}{dt}\right)^2 + \frac{1}{a^2}\right\} - \frac{2}{a}\frac{d^2a}{dt^2} \qquad . \quad . \qquad (3)$$

the units being such that the velocity of light and the constant of gravitation are unity. By (2) and (3)

$$\frac{6}{a}\frac{d^2a}{dt^2} = 2\lambda - 8\pi(\rho + 3p) \qquad . \quad . \quad . \qquad (4)$$

3. *Decay of Individual Motions.*—We shall regard a galaxy as being "at rest" if its angular co-ordinates (χ, θ, ϕ) remain constant. It is readily verified that its four-dimensional track will then be a geodesic. Observationally, galaxies "at rest" will appear to be receding from one another since the scale of the whole distribution is increasing. It is as though they were embedded in the surface of a rubber balloon which is being steadily inflated. Random individual motions relative to axes "at rest," whether of galaxies, of stars, or of atoms, constitute the pressure p in equation (3). By the usual kinetic theory p is $\frac{2}{3}\times$ kinetic energy per unit volume. To this should be added the pressure of radiation traversing space.

Lemaître obtains the striking result that when the universe expands the pressure changes *adiabatically*. The kinetic energy of the individual motions (analogous to heat) decreases by the amount of the work $p dV$ done by the pressure in the expansion of volume of the universe. (The expansion, however, is not *caused* by the pressure; we shall find below that expansion occurs even when $p = 0$.) It follows that as the

* I confine attention to expanding universes for this reason; but the equations are always such that time is reversible, and to every expanding solution there corresponds a contracting solution.

† For distances small compared with a we can write $r/a = \chi = \sin\chi$ approximately; and (1) becomes the usual expression for spherical polar co-ordinates (r, θ, ϕ).

universe expands the individual motions of the galaxies tend to decrease, not only relatively to the scale of the system, but absolutely. In fact the average random velocity changes proportionately to $1/a$. Thus, if the expansion during past history has been considerable, we may expect the spiral nebulæ to be nearly "at rest," so that the regular scattering apart will not be unduly masked by individual motions.

4. *Conservation of Mass.*—It simplifies the tracing of the course of expansion of the universe if we may assume that the total mass of the universe remains constant. This is not rigorously true. We may consider either the proper mass or the relative mass (*i.e.* mass relative to axes "at rest"). Unfortunately neither of these is strictly conserved :—

(1) Apart from radiation the proper mass is conserved ; the relative mass diminishes owing to the decrease of kinetic energy of random motion mentioned in § 3.

(2) In the conversion of matter into radiation relative mass is conserved ; the proper mass diminishes since radiation has no proper mass.

Thus both masses may diminish a little in the course of time. However, these are rather insignificant complications. In what follows we shall generally take $p = 0$, so that proper mass and relative mass are the same, and both will be conserved to our order of approximation.*

5. *Instability of Einstein's Universe.*—Setting $p = 0$ in (4) we have

$$3\frac{d^2a}{dt^2} = a(\lambda - 4\pi\rho).$$

For equilibrium (Einstein's solution) we must accordingly have $\rho = \lambda/4\pi$. If now there is a slight disturbance so that $\rho < \lambda/4\pi$, d^2a/dt^2 is positive and the universe accordingly expands. The expansion will decrease the density ; the deficit thus becomes worse, and d^2a/dt^2 increases. Similarly if there is a slight excess of mass a contraction occurs which continually increases. Evidently Einstein's world is unstable.

The initial small disturbance can happen without supernatural interference. If we start with a uniformly diffused nebula which (by ordinary gravitational instability) gradually condenses into galaxies, the actual mass may not alter but the equivalent mass to be used in applying the equations for a strictly uniform distribution must be slightly altered. It seems quite possible that this evolutionary process started off the expansion of the universe. Once started, it must continue to expand at an increasing rate. I have not, however, been able to decide theoretically whether the condensation ought to start an expansion rather than a contraction.

Alternatively we might suppose that the initial equilibrium was upset by the conversion of matter into radiation. Such a conversion does not change ρ (since the mass of the radiation is equal to that of the matter converted), but it increases p. From equation (4) we see that

* Questions connected with the pressure are treated very fully in a forthcoming paper by Professor de Sitter.

the increase of p gives a negative d^2a/dt^2; thus the world would start contracting. This explanation must accordingly be rejected. It seems likely on general grounds that the conversion of matter into radiation would not begin until after a considerable degree of condensation into galaxies had occurred and the consequent expansion had got well under way.

A paper by R. C. Tolman * has just appeared suggesting the conversion of matter into radiation as an explanation of the recession of the nebulæ. It is clear, however, that it cannot be an "explanation," since by (4) it tends to retard the expansion. Tolman, in order to simplify the problem, imposes an artificial condition equivalent to $d(\log a)/dt$ = const. I think that what he determines in his paper is not a rate of conversion of matter which would "account for" the observed value of $d(\log a)/dt$, but a rate which would prevent $d(\log a)/dt$ from increasing.

We may compare the conditions of equilibrium (*a*) when the world contains only matter at rest, and (*b*) when it contains only radiation. Setting da/dt, $d^2a/dt^2 = 0$ in (2) and (3) we have

$$8\pi\rho = -\lambda + 3/a^2, \qquad 8\pi p = \lambda - 1/a^2,$$

so that

$$1/a^2 = 4\pi(\rho + p), \qquad \lambda = 4\pi(\rho + 3p).$$

(*a*) For matter at rest, $p = 0$. Hence

$$a^2 = 1/\lambda.$$

(*b*) For radiation, $\rho = 3p$. Hence

$$a^2 = \tfrac{3}{2}/\lambda.$$

The total masses are found to be

$$(a)\ \frac{\pi}{2}\lambda^{-\frac{1}{2}}, \qquad (b)\ \frac{\pi}{4}\left(\frac{3}{2}\right)^{\frac{3}{2}}\lambda^{-\frac{1}{2}}.$$

The difference of mass in these two extreme cases is only 8 per cent.

6. *Rate of Expansion.*—Taking $p = 0$, let

a_e, M_e be the radius and mass of an Einstein universe,
a, M the radius and mass of the system under consideration.

We recall that a is a function of the time and the other three quantities are constants. Results for Einstein's universe are

$$\frac{2}{\pi}M_e = a_e = 1/\sqrt{\lambda} \qquad . \quad . \quad . \quad . \qquad (5)$$

The whole volume of the spherical world is $2\pi^2a_e^3$; hence the density $\rho_e = 1/4\pi a_e^2 = \lambda/4\pi$, as already found. The mass is here measured

* *Proc. Nat. Acad. Sci.*, **16**, 320 (April 1930). Lemaître in his concluding paragraph seems also to entertain the same view.

in the gravitational unit, which is such that the sun's mass is approximately 1·5 kilometres. By (2)

$$\left(\frac{da}{dt}\right)^2 + 1 = \tfrac{1}{3}a^2(\lambda + 8\pi\rho)$$
$$= \tfrac{1}{3}a^2\lambda + 4M/3\pi a.$$

Hence

$$\frac{da}{dt} = \sqrt{(\tfrac{1}{3}a^2\lambda - 1 + 4M/3\pi a)} \quad . \quad . \quad . \quad (6)$$

Three cases arise:—

(*a*) If $M > M_e$, the right-hand side does not vanish for any positive value of a, and the system can expand continuously from very small to very large radius. The minimum of da/dt is given by differentiating (6) with respect to a,

$$\tfrac{2}{3}a\lambda - 4M/3\pi a^2 = 0,$$

so that $a^3 = 2M/\pi\lambda$, or using (5)

$$a/a_e = \sqrt[3]{(M/M_e)}.$$

As the radius increases through this value of a, the rate of expansion slows down and increases again. The difficulty of applying this case is that it seems to require a sudden and peculiar beginning of things.

(*b*) If $M < M_e$, the right-hand side vanishes for two positive values of a, say R_1, R_2, and is imaginary for values of a, intermediate between R_1 and R_2. Hence either the world starting with a finite velocity of expansion expands to radius R_1 and then contracts again, or starting with a finite velocity of contraction contracts to radius R_2 and increases again. It is difficult to find a natural starting-point for the actual universe on the locus mapped out for it. If we decide to accept case (*b*) we should presumably assume that the actual universe began with radius R_2, so that initially $da/dt = 0$; since then it has continually expanded.

As $a \rightarrow R_2$, $da/dt \rightarrow 0$ like $\sqrt{(a - R_2)}$, and it follows that the radius remains in the neighbourhood of R_2 for a finite time only. With any reasonable estimate of the present degree of expansion, the date of the beginning of the universe is uncomfortably recent.

(*c*) The limiting case is when $M = M_e$ as in § 5. Then R_1 and R_2 coalesce to the value a_e. As $a \rightarrow a_e$, $da/dt \rightarrow 0$ like $(a - a_e)$; hence the time that the radius remains in the neighbourhood of a_e is logarithmically infinite. There is at least a philosophical satisfaction in regarding the world as beginning to evolve infinitely slowly from a primitive uniform distribution in unstable equilibrium. Hence this case is the most attractive. But in physics logarithmic infinities are usually deceptive, and when adapted to practical applications turn out to be not so very large. I do not think we prolong the effective age of the galaxies very much by accepting $M = M_e$ rather than case (*b*). We allow evolution an infinite time to get started; but once seriously started its time-scale of progress is not greatly different from case (*b*).

When $M = M_e$, (6) can be integrated without elliptic functions. The result, which is rather complicated, will be found in Lemaître's paper (equation (30)).

7. *Seeing Round the World.*—A question, debated perhaps more for its picturesqueness than for its practical importance, is whether we can see right round the universe. It is well known that this is possible in Einstein's world, but not in de Sitter's world. Lemaître's expanding universe is of intermediate type, and curiously the answer also is rather intermediate: we can see all parts of the world, but it is not likely that we shall be seen from all parts of the world.

A light-track is given by setting $ds = 0$ in (1). Taking ourselves to be at the origin $\chi = 0$ at time t_0, we are on a light-track given by

$$0 = -a^2 d\chi^2 + dt^2,$$

so that for the past (events at χ, t seen by us now)

$$\chi = \int_t^{t_0} \frac{dt}{a} \quad . \quad . \quad . \quad . \quad (7)$$

and for the future (an observer at χ, t who sees us as we are now)

$$\chi = \int_{t_0}^{t} \frac{dt}{a} \quad . \quad . \quad . \quad . \quad (8)$$

Since in the past a was always less than at present, χ in (7) increases indefinitely as $t \to -\infty$; so that we can see many times round the world, if the world has endured long enough. In the future a will continually increase at an increasing rate, and it is possible that the integral (8) may tend to a finite limit less than 2π as $t \to \infty$. Lemaître, using estimates from astronomical data, calculates that this actually is the case. Observers whose co-ordinate χ is beyond this limiting value can never see the events now happening in our galaxy; they see the earlier events, but their infinite lifetime only suffices to follow our history up to a certain moment at which the rate of expansion of the radius of the world became too rapid.

To put it another way—the velocity of light in angle (*i.e.* fraction of the whole circuit round spherical space) decreases as the radius of space increases. At first when the radius is small a light-wave can make a number of complete circuits, but ultimately it can only approach asymptotically a fixed position.

Out of curiosity I have calculated by Lemaître's formulæ the time when circumambulation by light ceased to be possible. This happened very early in the expansion when a had only reached the value $1{\cdot}003a_e$. (This applies to case (*c*).) When the expansion reached this amount every light-wave then existing entered on its last lap; waves that started later can never complete a circuit. I may add that in so-called "elliptical space" a short cut is provided which halves the distance to be traversed; by cheating in this way the light can return to its starting-point if it starts not later than the moment when $a = 1{\cdot}073a_e$.

8. *Spectral Shift.*—Taking ourselves to be "at rest" at $\chi = 0$

and an object seen by us to be "at rest" at some constant χ, we obtain by varying the limits of the integral in (7),

$$0 = \frac{\delta t_0}{a(t_0)} - \frac{\delta t}{a(t)}$$

where $a(t)$ is the radius of the world at time t. Hence

$$\frac{\delta t_0}{\delta t} = \frac{a(t_0)}{a(t)} \quad . \quad . \quad . \quad . \quad (9)$$

that is to say, the ratio of the period observed to the period emitted is equal to the ratio of the radius of the universe at the time of observation to the radius at the time of emission. This determines the amount of red-shift of the spectrum.

We can immediately find the present rate of expansion of the universe. The average red-shift of the spectra of spiral nebulæ amounts to about 500 km. per sec. per megaparsec, or about $\frac{1}{2000}$ of the velocity of light for a million light-years distance. Hence we set $\delta t_0/\delta t = 2001/2000$ in (9) for an interval $t_0 - t = 1{,}000{,}000$ years. Hence the radius of the universe has expanded by 1 part in 2000 in the last million years.

The result is impressive. It indicates that the radius of space has doubled within ordinary geological time. Let

$$\frac{1}{a}\frac{da}{dt} = \frac{1}{A}.$$

Then the foregoing result gives

$$A = 2 \cdot 10^9 \text{ light-years.}$$

This gives a hint of the scale of the phenomenon that we are concerned with. By (5) we can calculate that a (fictitious) Einstein world * of radius A would have a mass $2 \times 10^{22} \odot$ and a density

$$\rho_A = 3 \cdot 10^{-28} \text{ gm./cm.}^3$$

Remembering that the density in the neighbourhood of the sun is of order 10^{-24} or at the most 10^{-23}, and allowing for large comparatively empty regions between the galaxies, it is probable that the density ρ of the actual universe is considerably less than ρ_A. This conclusion will be of use in § 9.

If we attempt to look a long way back into the past, as in the problem of "seeing round the world", the ratio $a(t_0)/a(t)$ will probably be large, and what was originally visual light may be shifted into the infra-red. There is thus in an expanding universe a kind of gradual obliteration of the past through fading into longer and longer wave-lengths. There

* There is only one possible radius for an Einstein world (with $p = 0$) which is at present unknown. We are here calculating (for purposes of comparison) what would be the density if the unknown radius turned out to be A. We shall find later that it is actually less than A.

is a well-known speculation concerning "ghosts" of stars or nebulæ formed by reconvergence of the emitted light after passing round the world. (This is fanciful because the actual irregularity of the universe would be almost sure to spoil the convergence.) These ghosts become redder with age, and disappear out of our visual range.

9. *Numerical Estimates.*—Even if we assume for the theoretical reason referred to in § 6 that $M = M_e$, a further astronomical datum besides A is necessary in order to determine a. The required datum is an estimate of the actual mean density of the world. At present it is not possible to make an estimate that would be of any serious value.

Let $a/a_e = q$, so that q is a measure of the expansion that is supposed to have occurred since the original equilibrium condition collapsed. For $M = M_e$ we obtain from (5) and (6)

$$\frac{da}{dt} = \sqrt{(\tfrac{1}{3}q^2 - 1 + 2/3q)},$$

so that

$$\frac{1}{A^2} = \frac{1}{a^2}\left(\frac{1}{3}q^2 - 1 + \frac{2}{3q}\right) \quad . \quad . \quad . \quad (10)$$

Also

$$4\pi\rho = \frac{2M_e}{\pi a^3} = \frac{1}{a^2 q}.$$

Hence

$$\frac{\rho_A}{\rho} = \frac{1}{4\pi\rho A^2} = \frac{1}{3}q^3 - q + \frac{2}{3} \quad . \quad . \quad . \quad (11)$$

For our expanding universe $q > 1$, and it follows from (11) that $\rho_A > \rho$. It is satisfactory that this accords with our observational conclusion in the last section. If we are right in believing that ρ is *considerably less* than ρ_A, q will be large and (11) and (10) become approximately

$$\frac{\rho_A}{\rho} = \frac{1}{3}q^3 ; \qquad \frac{a^2}{A^2} = \frac{1}{3}q^2 \quad . \quad . \quad . \quad (12)$$

Hence

$$a_e^2 = \tfrac{1}{3}A^2 \quad . \quad . \quad . \quad . \quad (13)$$

so that

$$a_e = 1200 \text{ million light-years.}$$

Estimates of ρ, such as that made by Hubble, may perhaps be expected to be valid to within a factor 100. The corresponding determination of q and a is uncertain to a factor $(100)^{\frac{1}{3}}$. Thus our knowledge of the present radius of space remains very vague.

We conclude that the radius of space was originally about 1200 million light-years, that it has since expanded considerably, but to an amount practically undeterminable, and that its present rate of expansion is 1 per cent. in about 20 million years—a rate which will continue indefinitely.*

* It is seen from (6) that $\frac{1}{a}\frac{da}{dt}$ tends to a fixed limit $\sqrt{\tfrac{1}{3}\lambda}$ as $a \to \infty$.

10. *Miscellaneous Questions.*—De Sitter's world corresponds to $\rho = 0$, and is the limit approached as the density diminishes through expansion of volume of space. Accepting case (*c*), $M = M_e$, the history of expansion of the universe resolves itself into a gradual transition from Einstein's to de Sitter's world. According to § 9 the expansion has now proceeded so far that the de Sitter model gives much the better approximation; but this depends entirely on the truth of our estimate that the actual mean density of the universe is much less than 10^{-28} gm./cm.^{3} Admitting this, no great change is required in current theories which have assumed a de Sitter universe, provided that they do not concern themselves with early history.

De Sitter, however, used a different reckoning of space and time and "at rest" from that used in (1), so that the same phenomena were described by him in a different way. In particular he introduced "the slowing down of time" at great distances from the origin, which does not occur in the new formulæ. The present description involves fewer paradoxes and is undoubtedly easier to apprehend—as several investigators had already discovered. It has, moreover, the advantage that we now approach de Sitter's world as the limit of a series of worlds of gradually diminishing density; whereas formerly we had to start with a completely empty world, and very cautiously put a few material bodies into it.

It was originally urged against Einstein's world that it restored to a certain extent the absoluteness of space and time. The same applies to Lemaître's expanding universe in which there is a natural definition of "at rest." There is, however, no real conflict with relativity. The existence of a unique standard of reckoning is due to the artificial condition that we have imposed, viz. the complete homogeneity of ρ and p; for a universe which is not perfectly spherical the standard no longer exists. The breakdown can be illustrated as follows. In a perfectly spherical Einstein universe the rays of light from a star will, after going right round the world, converge to form a real image or ghost-star at a point P, the star itself having meanwhile moved to a point P′. In general P′ will not coincide with P; in fact after emitting the light the star may have burst into fragments which have scattered to a number of different points P′. We regard PP′ as an absolute motion of the star, and accordingly define a body "at rest" as one which coincides with its own ghost. If the world is not perfectly spherical the focussing of the reconverging rays will be imperfect, and the image will be diffused over a large area. The ghost no longer defines a definite point, and our definition of "at rest" fails. This is entirely analogous to absolute rotation; in a perfectly uniform world a quantity can be defined which with some justification may be called "absolute rotation"; in an uneven world the definition fails, but the deviation from Euclidean geometry in the actual universe is so slight that absolute rotation can be retained as an approximate conception.

For general qualitative reasoning it is useful to remember that the λ term in the gravitational equations is equivalent to a repulsive force from the origin varying directly as the distance, and the phenomena depend essentially on whether this repulsion does or does not prevail over the ordinary gravitational attraction of the matter present. Except in a closed space the repulsion must prevail at sufficiently great distances. It is only when we have to deal with a region so large that its total curvature amounts to a considerable fraction of a sphere that this simple outlook begins to fail, mainly because new questions arise as to the conventional definitions of distance, simultaneity, etc., when the region can no longer be treated as approximately flat.

11. *The Time-scale.*—The rapid expansion of the universe now proceeding is evidently unfavourable to a long time-scale of billions of years. It is difficult to produce a decisive objection, but there is a general incongruity. We cannot calculate how long a period elapsed from the disturbance of Einstein equilibrium by the beginnings of evolutionary development until the deviation from equilibrium reached a serious amount; but from the time when the universe had reached, say, 1·5 times its initial radius to the present day, it is scarcely possible to allow more than 10^{10} years. If the sun has really existed as a star for 5 billion years, it is odd that it should have waited so long and then formed its system of planets just at the time the universe toppled into a state of dispersion.

The universe is now doubling its radius every 1400 million years, and this rate will, if anything, slightly increase in the future. In 10^{10} years the spiral nebulæ will be 10 magnitudes fainter than they are now. With a time-scale of billions of years, astronomers must count themselves extraordinarily fortunate that they are just in time to observe this interesting but evanescent feature of the sky.

This last conclusion is not dependent on the theory of spherical space; it rests on the present observational evidence that (apart from three or four near nebulæ) the spirals are receding at a rate which must take them outside our range of observation in a comparatively short time, and that none are coming in to replace them. Unless a theory is invented which provides some force opposing this recession, there is no evading the rapid departure of nebulæ from our neighbourhood.

12. *Conclusion.*—The proof of the instability of Einstein's model greatly strengthens our grounds for interpreting the recession of the spiral nebulæ as an indication of world curvature. When this explanation was first suggested by de Sitter, all that was known was that of the two models known to satisfy Einstein's accurate equations one would exhibit a phenomenon of this kind. Now the position is that in any world that satisfies Einstein's equations, such a phenomenon (or its opposite, *i.e.* a general velocity of approach) must necessarily appear in the course of time; the phenomenon is not merely consistent with theory, but is foretold by theory. Of course it is possible that the recession of the spirals is not the expansion theoretically predicted; it might be some local peculiarity masking a much smaller genuine expansion; but the temptation to identify the observed and the predicted expansions is very strong.

The investigation is incomplete in that we have only been able to study a system of galaxies strewn all over the world. It would be desirable to supplement this by considering cases in which the material system is confined to a part of space; the problem presents greater mathematical difficulties, but these are perhaps not insuperable.

The theory gives a value of the total mass of the universe which ought to be fairly trustworthy. Since the mass initially formed an Einstein universe of radius 1200 million light-years (§ 9), it is found by the usual formulæ to be

$$1 \cdot 1 \,.\, 10^{22} \odot \;=\; 2 \cdot 3 \,.\, 10^{55} \text{ gm.}$$

It may be recalled that the only astronomical datum used in obtaining this value is the red-shift of the spectra of the nebulæ estimated at 500 km. per sec. per megaparsec. It is also postulated (1) that $M = M_e$, since this condition is necessary if the universe is to have a natural beginning in accordance with general evolutionary ideas; (2) that the mean density of matter throughout the universe is considerably less than 10^{-28}.

It may be noted that this gives a total of $1 \cdot 4 \,.\, 10^{79}$ protons in the universe. Another way of reckoning gives half this number, as it is possible that one-half of the spherical world is merely a mathematical duplication of the other half.

The Masses, Luminosities, and Effective Temperatures of the Stars. Second Paper. By E. A. Milne, F.R.S.

1. Though I have studied with great care Professor Eddington's two papers in *Monthly Notices*, 1930 January, I can find nothing in them which requires me to withdraw or modify any of my paper on "The Masses, Luminosities, and Effective Temperatures of the Stars" in *Monthly Notices*, 1929 November. Professor Eddington has misunderstood the range of ideas developed in my paper, and he nowhere makes real contact with my arguments. As I stated at the beginning of my paper, my approach to the subject is totally different from Professor Eddington's, but he completely ignores the consequences of the changed point of view.

In his first paper Eddington replies to an accusation I did not make, and in his second paper he makes a serious mistake in reversing some of my algebra, and so overlooks my main theorem.

Professor Eddington says: "he (Milne) believes that the photospheric opacity is a leading factor in determining the luminosity." I neither said nor implied any such thing. The luminosity of a star in a steady state is simply equal, and always equal, to the total source strength, whatever the surface opacity, or for the matter of that, the interior opacity. I said and proved that the rate of cooling of a star of given mass and *given relative density-distribution* * depends on its surface opacity. To assume a steady state it must adjust either its

* *I.e.* belonging to a given homologous family.